Strategies to teach proof: Promoting logical reasoning in primary school via combinatorial games

Emmanuel Beffara[1] and Luigi Bernardi[2, 3]

[1]Univ. Grenoble Alpes, CNRS, Grenoble INP, LIG, 38000 Grenoble, France
[2]Università di Roma Tre, Italy
[3]NRD, Dipartimento di Matematica Università di Bologna

Research in logic reveals a deep structural and behavioral equivalence between strategies in combinatorial games and formal proofs. Combinatorial games also offer a rich and accessible setting for exploring mathematical reasoning. Drawing on this formal correspondence, we discuss the specific logical difficulties primary school students encounter when finding and presenting strategies during a teaching experiment on two-player games. We then compare these findings with the challenges reported in the literature on proof among high school and university students, specifically investigating whether working with game strategies exhibits the same cognitive patterns as formal proving. Ultimately, our broader research project aims to determine whether leveraging this correspondence through carefully designed games can effectively promote the development of students' proof skills.

Keywords: Combinatorial games, Winning strategies, Mathematical proof, Primary school

Theoretical context

Mathematical proof and argumentation

In mathematical practice and education, determining the status of statements is a central point: is such statement true, false, or contingent (its truth depends on something), and why is it so. In this perspective, proof plays several roles (de Villiers, 1990) including "verification (concerned with the truth of a statement), explanation (providing insight into why it is true)" as well as "discovery (the discovery or invention of new results)" and "communication (the transmission of mathematical knowledge)". Indeed, the process of proving serves to explore and understand the meaning of a statement as well as convincing oneself and others of its status. This role in convincing is fundamentally related to argumentation, although the relationship between proof and argumentation is subtle (Durand-Guerrier et al., 2012). Mathematical argumentation appeals to logical deduction as well as to intuition and analogy, that is to both syntactic and semantic considerations. This has raised questions on whether the use of informal argumentation is an obstacle or a helpful precursor in learning proof, since it has been argued that they rely on different linguistic practice and have different epistemological foundations (Balacheff, 1999; Duval, 1991). Nevertheless, the dialectical nature of argumentation has been used as a foundation for didactic situations for the learning of proof, in particular from a problem-solving perspective (Arzarello & Soldano, 2019; Grenier & Payan, 2003). Moreover, game-theoretical interpretations of logic, notably following Hintikka and Sandu (1997)[1], have been used as a framework to study argumentation in mathematics classroom (Barrier et al., 2009).

Games for proving *vs* Games for teaching

Relationships between proofs and abstract games are well-established in mathematical logic. In Lorenzen and Lorenz (1978)'s interpretation, mathematical statements induce formal games between a proponent and an opponent, with winning strategies corresponding to proofs. The Curry–Howard

[1]Although the cited handbook dates from 1997, the underlying ideas go back to Hintikka's work in the late 1950s.

isomorphism (Howard, 1980) further relates proofs with programs seen as computational strategies. A key point is that simple rules are introduced without a prescribed logical interpretation and focus is on the interactive structure of proofs (seen as strategies) (Girard, 2011). Consequently, in the absence of a model in which to establish truth values, the sole relevant aspect is consistency within the interaction. In other words, without a semantics in the classical sense, arguments both for and against a given statement may be accepted; rather than being bound by constraints of truth and falsity, they must simply guarantee adherence to the rules of interaction. Therefore, a formula derives its meaning (*i.e.*, a semantics) exclusively through the game dynamics; this is exactly why it is referred to as *Game Semantics*. Within this theoretical framework, effective strategies, which are computable and communicable, are the formal objects that stand as candidates for being considered as proofs.

Teaching formal proof rules is often ineffective for novices, for lack of perceived meaning, even in the context of transition to university level (Selden, 2012). Besides, evaluating whether arguments are valid, even without reference to formal proof rules, requires to consider proofs as objects of study, as a kind of meta-reasoning, which is even more challenging. However, following rules in games is part of every child's practice, and reasoning about strategies, when properly encouraged, emerges naturally within a ludic context. Moreover, the introduction of games into the mathematics curriculum is supported by strong pedagogical reasons: students are engaged, immerse themselves in activities, discuss solutions, and analyse different strategies (de Vries et al., 2021). The game allows teachers to put students in problem-solving situations, functional to the ability to develop strategies and to unleash potential that students are sometimes unaware of and that are seldom highlighted in standardised situations. This approach has a positive impact not only cognitively but also socially, emotionally, and linguistically.

Formal proofs, strategies, and game design

The formal correspondence between games and proofs serves as our epistemological foundation. Both objects are conceived as abstractions of planning an interaction against an opponent, where validation is determined by a notion of winning. Within this framework, we examine two central aspects of proving: developing a winning strategy and formulating it in a communicable and persuasive way. We hypothesize that reasoning about strategies in appropriately designed games engages the same cognitive processes as reasoning about proofs, and furthermore that this form of reasoning is accessible to young students, as it builds on the anticipatory thinking familiar to players of strategic games.

Delving deeper into the "game of proving", the process of validating a statement is a game between two players—traditionally known in the literature as ∃loise and ∀belard—who "play the roles" of the two quantifiers. In this interactive dynamics, the existential quantifier ($\exists$) acts as the active "Proponent", whose general rule is to identify a witness that *verifies* a proposition. Conversely, the universal quantifier ($\forall$) represents the "Opponent", which challenges the proponent by arbitrarily selecting values in an attempt to *falsify* the statement. These "rules of engagement" alternate, thereby addressing formulas with alternating quantifiers. Even in their most elementary forms, these structures have been employed to explore the intersection of logic and mathematics education (Dubinsky & Yiparaki, 2000). Over time, these models have evolved into more refined frameworks for the "game of proving". Today, most models in game semantics appear as asymmetric two-player combinatorial games without the possibility of a draw, where one player wins by driving the other into a contradiction, whereas the other player wins if the play is infinite.

The choice of games in an educational context, however, proves to be a delicate matter. Indeed, while the proof-theoretical approach is concerned with a formal notion of game whose meaning lies essentially in the image of two rational agents interacting according to some rules with a notion of winning and losing, games as a pedagogical device rely on a more concrete notion of game, with the intention of favouring engagement through the use of emotion and social interaction, a dimension which is out of scope in formal logic. It is thus necessary to choose games that are meaningful from a pedagogical and cultural standpoint. On the other hand, these games must capture complex logical aspects related to game semantics. The long-term aim of our educational path is to introduce students to such asymmetric, potentially infinite games. However, the didactical assumptions guiding the present study are that it is preferable to begin with symmetric rather than asymmetric games, and with finite games before addressing infinite ones.

Consequently, the present study focuses on the conceptually simpler game of Tic-tac-toe, which serves as a stepping stone toward the more general game of proving. Despite its simplicity, Tic-tac-toe captures relevant logical aspects that can be framed within the theoretical context described above. In particular, given the limited number of possible moves, it is feasible to construct a game tree to represent a strategy, as we shall see in the following section. To bring these aspects to light, merely allowing the class to play the game is insufficient; rather, it is necessary to design specific situations where the need for a strategy naturally emerges.

The broader educational trajectory—which falls outside the scope of this paper—progresses through increasingly complex games, ultimately culminating in a adaptation of the game of proving for propositional logic. In this final game, if the Proponent has a winning strategy for a given formula, this strategy is equivalent to a proof of that formula. In other words, the games within the instructional sequence aim to reveal that games inherently possess a logical structure, and that this structure shares distinct dynamic parallels with the process of proving. In this phase, we are going from games to logic. As the logical structure of the activities grows more complex, the sequence gradually approaches the proof-theoretic model. By the time the final game is reached, the perspective is inverted: proofs themselves can now be interpreted as games, thereby going from logic to games.

Research questions and method

Based on these notions, our aim is to investigate to what extent the correspondence between proofs and strategies extends to associated skills in proving and formulating strategies, and what it implies in terms of transferability of skills. We focus on the following research questions.

Q1: What logical difficulties do primary school students have in finding and presenting strategies in combinatorial games?

Q2: Does work on strategies show the same patterns and challenges as work on proofs?

We approach these questions through the methodology of didactic engineering (Artigue, 2015). Specifically, we designed a didactic sequence situated within a ludic context, featuring tasks that require students to elaborate precise reasoning and discourse. The correspondence between proofs and strategies, as outlined in the previous section, serves as the theoretical framework for designing these situations and linking the tasks to mathematical proof.

Regarding Q1, our focus lies on the elaboration and formulation of strategies rather than on justifying their formal validity, as the latter requires a higher level of abstraction. This represents the initial step in conceptualizing proofs and strategies—similar to how students learn to write and test computer programs long before formally proving their correctness. To answer this question, we rely on a qualitative analysis of two primary data sources: the students' written productions and the teacher's field notes gathered during the experiment.

Regarding Q2, we compare the difficulties students encountered during the combinatorial two-player games with those widely reported in the mathematics education literature on proof. Particular attention is given to typical challenges, such as exhaustiveness in case analysis and the proper handling of universal and existential statements. To address this question, our analysis triangulates the findings emerging from Q1, the students' specific written productions (with a particular focus on their graph representations), and the existing literature on proof-related difficulties.

Ultimately, our overarching qualitative analysis focuses on identifying and examining instances where the theoretical framework bridging games and proofs provides a meaningful lens to interpret the students' work and their emerging mathematical reasoning.

Design and a priori analysis

In our educational path, named *Lovleis*, the class reads a story in which they themselves are the protagonists, travelling through various locations of a fairy-tale world[2]. Each location corresponds to a two-player game for the class to play and study, presented in order of increasing complexity. In this paper, we focus on the first part of the sequence. The students enter a fairy-tale world and arrive in Tictacto, a city of tic-tac-toe enthusiasts. After briefly reviewing the rules and playing a few games, they take part in a tournament designed to assess their understanding of the rules and to begin eliciting implicit strategies. After the tournament, the class moves to the explicit analysis of strategies through an *escamotage*: Timoteo, a resident of Tictacto, appears in the story and asks the class for help with a problem of his: he can never manage to win at tic-tac-toe against his sister Celeste, who always goes first and places the "X" in the center of the grid. Working in pairs, the students write a letter to Timoteo to suggest how he can avoid losing to his sister. As can be observed, to simplify the class's analysis, we restrict attention to the subset of games that begin with an "X" in the center. The students' productions that will be analysed in the following section were produced at this stage.

Two major conceptual challenges stand out as both sources of difficulty and essential learning objectives for students. First, a strategy must clearly communicate the sequential nature of play, specifying the order of moves. Second, it must be sufficiently general: since a player cannot predict an opponent's moves in advance, the strategy must account for all possible responses, ensuring a counter for every opponent action. Crucially, students must grasp that a strategy is developed *a priori*, before the game begins. From a computational standpoint, this mirrors the structure of a program, guiding the player's moves in response to the opponent's actions. The kind of planning involved in constructing a strategy is precisely analogous to that required in designing algorithms (Rogalski & Samurçay, 1990). It is thus expected that students will first produce particular plays or partial strategies, which are analogous in the game context to examples (more or less generic) or incomplete arguments in a proof context.

After discussing the strategies proposed by the students, a whole-class discussion is initiated to identify a graphical representation of the notion of strategy that is straightforward to interpret and highlights both its sequential nature and its generality. The strategy is eventually represented as a tree as in Figure 1: positions are nodes, linked by arrows indicating moves. Conventionally, Celeste has the "X" and Timoteo the "O". When it is Timoteo's turn, there is a single arrow indicating the move that Timoteo must make. On the other hand, when it is Celeste's turn, all the moves that Celeste could make during an actual game are considered. This way, Timoteo's strategy is capable of responding to any move made by Celeste. At this point in the sequence, students are provided with a worksheet containing an incomplete graph; they are asked to complete it and to extend a subtree of their choice.

Realisation and observation

The path was partially implemented in 3rd and 4th grade classes in Italian primary schools (41 students in total) and fully implemented in three 5th grade classes in an Italian school in Colombia (63

[2]The full story is available in Italian at https://www.oiler.education/scuola/materiali/primaria/lovleis

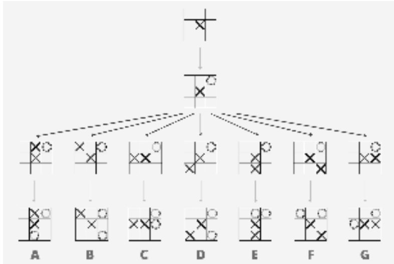
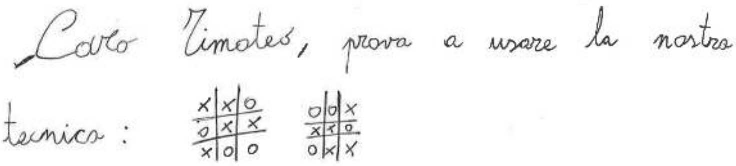


Figure 1: A partial strategy for Timoteo. To complete it, all the sub-trees A, B, C, D, E, F, G should be completed.



"Dear Timoteo, try and use our technique."

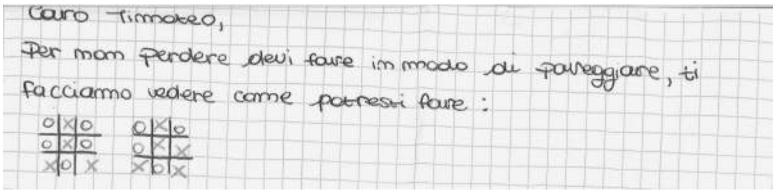
Figure 2: The students only show two final positions where the game ends in a draw

students) in 2023 and 2024[3].

In the first activity, students are asked to write a letter to help Timoteo stop losing at Tic-Tac-Toe against his sister Celeste. Almost all students first struggled with effectively communicating the *sequentiality* of the strategy, as exemplified in Figure 2. Indeed, the final position of a play does not indicate the order in which the moves were made, so Timoteo cannot know which move to make at what time. We observed the same error during an experiment in a final-year high school class, as shown in Figure 3: if not explicitly addressed, this difficulty can persist even among students who have already worked on mathematical proofs (albeit not very successfully). After discussing the problem with the students, a solution many found was to assign numbers to the symbols to indicate at what point in the play each move is executed, as shown in Figure 4. At this stage, a crucial issue becomes explicit: *how can one be sure of the move Celeste will make?* Most students, if not all, overlooked the generality of the strategy, believing they could predict Celeste's moves to guide Timoteo's strategy, as exemplified by the representative case in Figure 5.

In other words, most students had difficulty considering *all* the potential moves Celeste could make at each turn. Anticipating a point to which we will return in more detail later, when a strategy is specified in a two-player game, the moves of the player for whom the strategy is written are quantified by the existential quantifier $\exists$ (*i.e.*, the strategy tells this player which move to make), whereas the opponent's moves are quantified by the universal quantifier $\forall$ (*i.e.*, the strategy must work for any moves the opponent may choose). From a logical standpoint, the difficulty concerning generality can be seen as taking both the player's moves and the opponent's moves to fall under an existential quantifier. The students—although understanding that their approach lacked generality—were unable to do anything but address specific plays (*i.e.*, examples), as shown in the letter in Figure 6. In their letter, the students also provide what can be termed a *partial strategy*: if one manages to set up a "fork", then one can win in the next turn. In these cases, where it seems clear that Celeste's move cannot be predicted, the approach is limited to providing examples, thus losing generality. Returning to the logical perspective, the quantifier associated with Celeste's moves can now be viewed, in a sense, as a hybrid between an existential and a universal quantifier, as illustrated in Figure 7.

[3]The authors extend their thanks to teacher Giorgia Damiano for conducting the activities in Colombia.



"Dear Timoteo, in order not to lose you have to force a draw; we'll show you how you could do that:"

Figure 3: Production of a student from a final-year high school class in Italy.

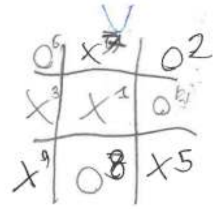


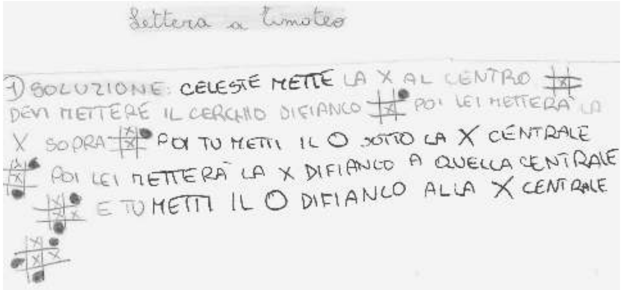
Figure 4: The students added indices to indicate the sequence of moves.

In summary, two profound conceptual challenges that cause difficulties were observed, in accordance with the *a priori* analysis The first challenge is understanding that in describing a strategy, it is essential to express the sequence of moves leading to a certain situation, emphasising its relevance. The second one is realising that a strategy must be general in nature, considering all possibilities, as one cannot predict the opponent's moves. The workaround found by students who understood that Celeste's moves could not be predicted, but did not manage to formulate a satisfactory reasoning, was to provide examples or partial strategies. It should be noted that the second challenge becomes apparent only after addressing the first one.

Analysis

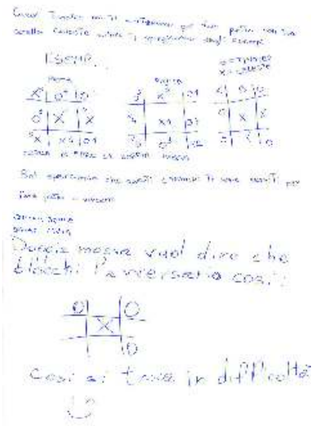
To explore the relationship between quantifiers and strategies more deeply, let us imagine a two-player game with only two moves: the opponent, *O*, plays first, followed by the player, *P*, for whom the strategy is being specified. Having a winning strategy in this game for player *P* means that, regardless of the move made by *O*, there exists a counter-move (that is, a move which depends on *O*'s choice) that *P* can make in order to win. In mathematical terms, having a winning strategy for *P* in this scenario means satisfying the formula $\forall O_{\text{move}}\,\exists P_{\text{move}}(P \text{ wins})$. Similarly, having a non-losing strategy (*i.e.*, one that guarantees either a win or a draw) corresponds to satisfying the formula $\forall O_{\text{move}}\,\exists P_{\text{move}}\neg(O \text{ wins})$. On the other hand, treating *P* and *O* symmetrically, merely providing a single example of a play, as in the production shown in Figure 5, amounts to satisfying the formula $\exists O_{\text{move}}\,\exists P_{\text{move}}\neg(O \text{ wins})$. In more refined productions, such as those in Figures 6 and 7, there is an awareness that the existential quantifier $\exists$ is not the appropriate one for *O*'s moves. In some cases, an attempt is made to replace it with an "intermediate quantifier"—one that is neither $\forall$ (since the strategy does not account for all possible moves), nor $\exists$ (since there is a recognition that a single example is not sufficient).

Let us now analyze the written productions of three different students regarding the graph-completion worksheet. In the production in Figure 8a, the student begins by correctly drawing the five arrows representing Celeste's five possible moves. They then realise that in four out of these five cases, Celeste fails to block Timoteo's imminent three-in-a-row, and therefore do not continue, assuming that Timoteo will win on the following turn. They proceed only with the case they rightly consider interesting—the one where Celeste blocks the three-in-a-row. However, at the next step, the student makes a "mistake" by drawing all of Timoteo's possible moves ($\forall$), whereas they should have chosen



"Solution: Celeste places the X in the centre, you need to put the circle next to it, then she will place the X above, and then you put the O below the central X. After that, she will place the X next to the central one, and you put the O next to the central X."

Figure 5: The student treats Timoteo and Celeste as equivalent.



"Dear Timoteo, we will help you to draw with your sister Celeste, so we will explain some examples to you.
[...]
Well, we hope these examples will help you to draw or win.
A double move means that you block the opponent like this, so they find themselves in trouble."

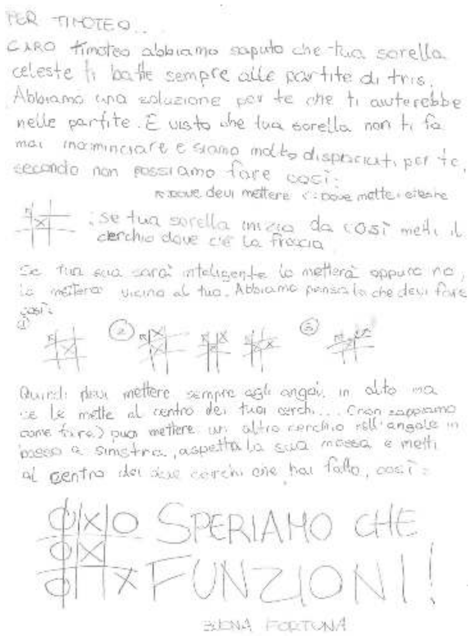
Figure 6: The students provide several examples, and then present a partial strategy.

only the winning move (∃). They continue solely with the correct move (in which Timoteo blocks the imminent three-in-a-row by Celeste), and then correctly indicate Celeste's three possible moves. In other words, rather than a real error, this can be interpreted as a written search for the best move. In any case, in this context we regard it as a sign of deeper understanding to introduce a superfluous universal quantifier rather than to omit an indispensable one.

In the second production, shown in Figure 8b, the student still confuses existential and universal quantifiers. It is important, however, to emphasise that this does not correspond to a poor understanding of the task at hand: a clear meaning is assigned to the graph (the two players alternate correctly at each level) and, crucially, the move considered at each level is the best one. In essence, the student has a good grasp of how the game unfolds and of its sequential structure, but targeted work is still needed on generalisation and, more broadly, on the universal quantifier.

In the third production, shown in Figure 8c, although only partial and not entirely clear, the task is nonetheless carried out correctly: the number of edges drawn is correct, but the desire to finish quickly prevails over careful and precise communication. For instance, when representing the final positions, the student does not depict the full position but only records the last move. However, the student's approach is highly significant because it ties directly back to our theoretical framework: rather than focusing on whether individual positions are winning states, the student is primarily interested in the *interaction* between the two players—and thus, the interaction between the two quantifiers.

In asking students to complete the graphs, we did not expect them to arrive immediately at a fully correct solution; on the contrary, the activity was designed precisely as an opportunity to isolate, understand, and discuss the main difficulties. It is therefore not surprising that most students made some errors in completing the task: these outcomes are entirely natural, given that this is only the first step in a broader educational trajectory.



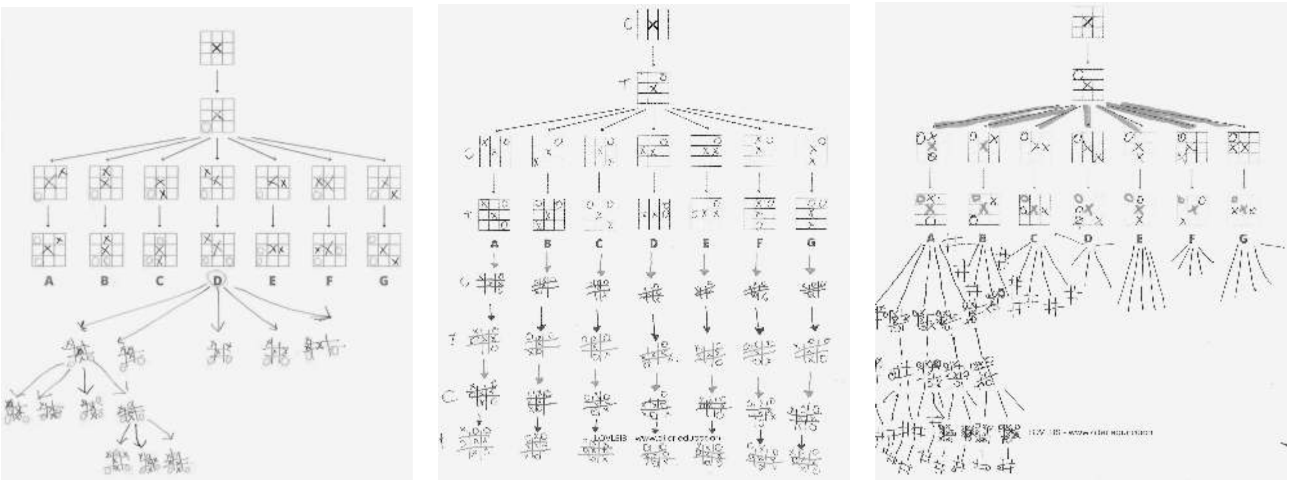
"Dear Timoteo, we heard that your sister Celeste always beats you at tic-tac-toe. We have a solution for you that would help you in the games. And since your sister never lets you start and we are very sorry for you, we think we can do like this: [...] if your sister starts like this, put the circle where the arrow is. If your sister is intelligent she will put it or not, she will put it next to yours. We thought that you have to do like this: [some possibilities are analysed]. So you must always put [your circle] in the top corners, but if she puts it in the middle of your circles... (we don't know what to do) you can put another circle in the bottom left corner, wait for her move and put [one] in the middle of the two circles you made, like this. WE HOPE IT WORKS! Good luck"

Figure 7: The student analyses some paths of the strategy tree, clearly not exhausting all possibilities.

Our further goal is to identify whether the difficulties observed are the same as those occurring in a context of proof elaboration for older students, following the formal correspondence between strategies and proofs. Selden and Selden (2015) report how some students, even after specific training, confuse empirical evidence with deductive proof. As discussed above, this difficulty is already present when talking about winning strategies in primary school: the examples attempt to serve as a surrogate for the lack of generality. The report by Selden and Selden continues by discussing how one source of students' difficulties in discerning the logical structure of theorems is a lack of understanding of the meaning of quantifiers and that their order matters (also reported in (Dubinsky & Yiparaki, 2000)): undergraduate students often fail to appreciate the importance of quantifier order, wrongly assuming that switching existential and universal quantifiers has no effect. These aspects also emerge clearly in our study: the lack of understanding of the order of quantifiers already manifests itself, implicitly, in students' difficulty in effectively communicating the sequential nature of a strategy. Moreover, and above all, students' difficulties with quantifiers are not only syntactic but primarily semantic, which highlights a much more foundational and significant issue.

This discussion of quantifiers is corroborated in a geometric setting, where it is typical to confuse a proof of a theorem of the form $\forall\,\exists$ with a single construction that satisfies the conditions of the theorem (*i.e.*, of the form $\exists\,\exists$); and this occurs even after specific training on proof (Chazan, 1993).

Morselli (2006) conducted interviews with university students, focusing on a specific case of proofs in algebra, and found that participants' argumentative processes could be classified into four profiles: (a) work exclusively through algebraic manipulation; (b) short explorations with examples and shift to algebraic proof; (c) extended explorations with examples leading to reasoning about the conjecture; (d) unfocused explorations with examples. She identified, in particular, that participants exhibiting the fourth argumentation profile were less successful than other students. This suggests that exploration with examples can be very productive for proving as long as the exploration is focused and purposeful. Let us note that, throughout our educational path, the explorations are always meaningful: the students, while playing, have the central goal of winning, which greatly fosters the development of conjectures and arguments. Similarly, Lin and Wu (2007) suggest that the features of given examples influence the kinds of generalisations that students make. In cases where students are asked to reason



(a) **(b)** **(c)**

Figure 8: A strategy tree completed partially but correctly (a) and two examples of students' productions (b) and (c).

from examples to prove a given conjecture, it may support students' argumentation process if a range of examples are provided for their review or if students are encouraged to generate their own examples so that they can determine which features are variant under the given conditions.

Conclusion

In conclusion, the analysis of this teaching sequence allows us to explicitly address our research questions. Regarding Q1, which explored the logical difficulties primary school students face in finding and presenting strategies, our findings reveal a significant gap between operational gameplay and explicit formulation. Although students often possess implicit strategies—as evidenced by their ability to play the games successfully—they struggle with determining what to communicate and how to articulate it. We identified two primary areas of difficulty: managing the sequentiality of the game (the necessity to clearly define the specific order of moves) and grasping its generality (the requirement to anticipate and account for all possible moves of the opponent, rather than predicting a single specific response). Regarding Q2, our theoretical framework allows us to confirm that working on game strategies exhibits the exact same cognitive patterns and challenges as working on formal proofs. The obstacles identified in Q1—specifically the struggle with sequentiality and generality—closely mirror the difficulties widely reported in the literature concerning proof construction among high school and university students. In both contexts, a central source of difficulty lies in handling alternating quantifiers. Ultimately, these findings suggest that engaging with strategies in combinatorial games serves as a highly effective pedagogical approach. It provides a rich, accessible environment to familiarize young students with the complex reasoning patterns and argumentative dynamics that are foundational to mathematical proof.

References

Artigue, M. (2015). Perspectives on design research: The case of didactical engineering. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education* (pp. 467–496). Springer. https://doi.org/10.1007/978-94-017-9181-6_17

Arzarello, F., & Soldano, C. (2019). Approaching proof in the classroom through the logic of inquiry. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 221–243). Springer. https://doi.org/10.1007/978-3-030-15636-7_10

Balacheff, N. (1999). Is argumentation an obstacle? Invitation to a debate. *Preuve*. http://www.lettredelapreuve.org/OldPreuve/Newsletter/990506Theme/990506ThemeUK.html

Barrier, T., Durand-Guerrier, V., & Blossier, T. (2009). Semantic and game-theoretical insight into argumentation and proof. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *ICMI Study 19 conference: Proof and Proving in Mathematics Education* (pp. 77–82). https://hal.archives-ouvertes.fr/hal-00804075

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, *24*, 359–387. https://doi.org/10.1007/BF01273371

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17–24.

de Vries, H. G., Polk, K. D., & Missall, K. N. (2021). Math talk during traditional and digital number board game play. *Journal of Applied Developmental Psychology*, *76*. https://doi.org/10.1016/j.appdev.2021.101312

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *CMBS issues in mathematics education* (pp. 239–289). American Mathematical Society.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education: The 19th ICMI study* (pp. 349–367). Springer. https://doi.org/10.1007/978-94-007-2129-6_15

Duval, R. (1991). Structure du raisonnement deductif et apprentissage de la demonstration [Structure of deductive reasoning and learning of proof]. *Educational Studies in Mathematics*, *22*(3), 233–261. https://doi.org/10.1007/BF00368340

Girard, J.-Y. (2011). *The Blind Spot: Lectures on Logic*. European Mathematical Society.

Grenier, D., & Payan, C. (2003). Situations de recherche "en classe" : essai de caractérisation et proposition de modélisation [Research situations "in the classroom": an attempt at characterization and a proposed model]. In V. Durand-Guerrier & C. Tisseron (Eds.), *Actes du séminaire national de didactique des mathématiques: année 2002* (pp. 189–203). ARDM.

Hintikka, J., & Sandu, G. (1997). Game-theoretical semantics. In J. van Benthem & A. ter Meulen (Eds.), *Handbook of Logic and Language* (pp. 361–410). North-Holland. https://doi.org/10.1016/B978-044481714-3/50009-6

Howard, W. A. (1980). The formulae-as-types notion of construction. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Academic Press.

Lin, M.-L., & Wu, C.-J. (2007). Uses of examples in geometric conjecturing. *Proc. 31st Conf. of the International Group for the Psychology of Mathematics Education*, *3*, 209–216.

Lorenzen, P., & Lorenz, K. (1978). *Dialogische Logik* [Dialogical Logic]. Wissenschaftliche Buchgesellschaft.

Morselli, F. (2006). Use of examples in conjecturing and proving: An exploratory study. *Proc. 30th Conf. of the International Group for the Psychology of Mathematics Education*, *4*, 185–192.

Rogalski, J., & Samurçay, R. (1990). Acquisition of Programming Knowledge and Skills. In *Psychology of Programming* (pp. 157–174). Elsevier. https://doi.org/10.1016/B978-0-12-350772-3.50015-X

Selden, A. (2012). Transitions and Proof and Proving at Tertiary Level. In G. Hanna & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education: The 19th ICMI Study* (pp. 391–420). Springer Netherlands. https://doi.org/10.1007/978-94-007-2129-6_17

Selden, A., & Selden, J. (2015). A comparison of proof comprehension, proof construction, proof validation and proof evaluation, Extended Abstract, KHDM Conference, Hanover, Germany, December 1-4, 2015. https://doi.org/10.13140/RG.2.1.4817.4805